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Quasi-Maxwellian Fields in Riemann–Cartan Spacetime

L. C. Garcia de Andrade[1,2]

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The electric part of the Weyl conformal curvature tensor is computed for a static, cylindrically symmetric cluster in Einstein-Cartan (EC) theory.

1. INTRODUCTION

The electric and magnetic (dual) parts of the Weyl conformal tensor have been studied (Novello and Salim, 1978; Novello and de Oliveira, 1980). Novello and Salim (1978) discuss the so-called quasi-Maxwellian equations of gravity, which are dynamical equations which relate the derivatives of the electric and magnetic parts of the Weyl tensor in terms of the expansion θ, shear $\sigma_{\mu\nu}$, and vorticity $\omega_{\mu\nu}$ of the fluid. However, these computations have been done in the context of the Riemannian V_4 spacetime manifold. Here I present a dynamical relation between the evolution parameters, θ, $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ and the electric part of the Weyl tensor: $E_{\mu\nu}=-C_{\mu\alpha\nu\beta}V^\alpha V^\beta$ in V_4 Riemann-Cartan spacetime.

2. THE RIEMANN-CARTAN GEOMETRY OF QUASI-MAXWELL FIELDS

Let us consider a Riemann-Cartan spacetime with Lorentz signature $(+---)$ endowed with an affine connection $\Gamma^\mu_{\alpha\beta}$, where (Tauber, 1988)

$$\Gamma^\mu_{\alpha\beta}=\left\{{\mu \atop \alpha\beta}\right\}-\kappa^\mu_{\alpha\beta} \tag{2.1}$$

where $\left\{{\mu \atop \alpha\beta}\right\}$ are the usual Christoffel-Levi-Civita symbols of general relativity and $\kappa^\mu_{\alpha\beta}$ is the contortion tensor

$$\kappa^\mu_{\alpha\beta}=S^\mu_{\alpha\beta}-S^\mu_{\beta\alpha}+S^\mu_{\alpha\beta} \tag{2.2}$$

[1]Department de Física Teórica, Instituto de Física, Universidade do Estado do Rio de Janeiro, CEP: 20550, RJ, Brazil.

[2]Present address: Department of Physics, University of Alabama, Huntsville, Alabama 35899.

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where $S^{\mu}_{[\alpha\beta]}$ is the torsion tensor obeying the Weyssenhoff fluid relations

$$S^{\mu}_{\alpha\beta} = V^{\mu} S_{\alpha\beta} \tag{2.3a}$$

$$S^{\mu}_{\alpha\mu} = V^{\mu} S_{\alpha\mu} = 0 \tag{2.3b}$$

The Weyl tensor in V_4 is given by

$$C_{\kappa\lambda\mu v} = R_{\kappa\lambda\mu\nu} - R_{\lambda[\mu} g_{v]\kappa} + R_{\kappa[\mu} g_{v]\lambda} + \tfrac{1}{3} R g_{v[\kappa} g_{\lambda]\mu} \tag{2.4}$$

where now the only properties of the Riemann tensor are $R_{\mu\nu(\rho\sigma)} = 0$ and the Ricci tensor has a skew-symmetric component $R_{[\mu\nu]}$.

Let us now consider the definitions of the electric and magnetic parts of the Weyl tensor

$$\mathbb{E}_{\mu\nu} = -C_{\mu\alpha\nu\beta} V^{\alpha} V^{\beta} \tag{2.5a}$$

$$\mathbb{H}_{\mu\nu} = -{}^{*}C_{\mu\alpha\nu\beta} V^{\alpha} V^{\beta} \tag{2.5b}$$

where * represents the dual of a tensor $f_{\alpha\beta}$ given by ${}^{*}f_{\alpha\beta} = \eta_{\alpha\beta\gamma\delta} f^{\gamma\delta}$, where $\eta_{\alpha\beta\gamma\delta} = (-g)^{1/2} \varepsilon_{\alpha\beta\gamma\delta}$, and $\varepsilon_{[\alpha\beta\gamma\delta]}$ is the totally skew-symmetric Levi-Civita tensor. Here I shall be concerned only with the electric part of the Weyl curvature in U_4, since to consider $H|_{\mu\nu}$ is just to define the dual * of the spin density tensor; this is just to take the definition ${}^{*}S^{\lambda}_{\alpha\beta} = \eta_{\alpha\beta\gamma\delta} S^{\gamma\delta\lambda}$ (Tauber, 1988). From expression (2.5a) and the Ricci identity in V_4,

$$(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}) V^{\lambda} = -R^{\lambda}_{\mu\nu k} V^{k} + (\kappa^{k}_{\mu\nu} - \kappa_{v\mu}) \nabla_{k} V^{\lambda} \tag{2.6}$$

one can easily deduce the following dynamical expression for $\mathbb{E}_{n\lambda}$:

$$\begin{aligned}
&(\dot{\omega}^{\lambda}_{v} + \dot{\sigma}^{\lambda}_{v} - \tfrac{1}{3} h^{\lambda}_{\nu} \dot{\theta}) - (\omega^{\lambda}_{\mu} + \sigma^{\lambda}_{\mu} - \tfrac{1}{3}\theta h^{\lambda}_{\mu}) \cdot (\sigma^{\mu}_{v} + \omega^{\mu}_{v} - \tfrac{1}{3} h^{\mu}_{v} \theta) \\
&= \mathbb{E}^{\lambda}_{\nu} - \tfrac{1}{2} R^{\lambda}_{\mu} V^{\mu} V_{v} + \tfrac{1}{2} R^{\lambda}_{\nu} + + \tfrac{1}{2}(R_{k\mu} V^{\mu} V^{k} - \tfrac{1}{3} R) \delta^{\lambda}_{\nu} \\
&\quad - \tfrac{1}{2} R_{kv} V^{k} V^{\lambda} + \tfrac{1}{6} R V_{\nu} V^{\lambda}
\end{aligned} \tag{2.7}$$

where the dot represents the absolute derivative $V^{\mu}\nabla_{\mu}$ in U_4, and θ, $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ are, respectively, the expansion, shear, and vorticity of the spinning fluid.

Let us apply equation (2.7) to the simple static, cylindrically symmetric cluster in EC where $T_{\mu\nu} = \rho V_{\mu} V_{\nu}$ and the only nonvanishing component of $S^{\gamma}_{\nu\mu}$ is $S^{0}_{12} = -S^{0}_{21}$. We also adopt the point of view of a comoving observer where $V^{\mu} = \delta^{\mu}_{0}$.

To simplify matters, we still consider a metric where $g_{00} = 1$. From the energy-momentum tensor of EC

$$T^{\mathrm{EC}}_{\mu\nu} = T^{\mathrm{GR}}_{\mu\nu} + \tfrac{1}{2} \nabla_{\alpha} S^{\alpha}_{\mu\nu} \tag{2.8}$$

and the EC equations

$$R_{\mu v} - \tfrac{1}{2} g_{\mu\nu} R = T^{\mathrm{EC}}_{\mu\nu} \tag{2.9}$$

it is easy to compute equation (2.7) for this particular static ($\sigma_{\mu\nu}=0$, $\omega_{\mu\nu}=\theta$) cosmological model in EC,

$$\mathbb{E}^{\lambda}_{v}=\tfrac{1}{4}R^{\lambda}_{0}V_{v}+T^{\lambda}_{0}V_{v}-\tfrac{1}{2}(T^{\lambda}_{v}+\tfrac{1}{3}R\delta^{\lambda}_{v}+\tfrac{1}{2}\nabla_{\alpha}S^{\alpha\gamma}v)+T_{00}\delta^{\lambda}_{v} \quad (2.10)$$

which has components

$$\mathbb{E}^{0}_{0}=\tfrac{1}{2}(\rho+\tfrac{1}{6}R) \quad (2.11a)$$

$$\mathbb{E}^{1}_{1}=\mathbb{E}^{2}_{2}=\mathbb{E}^{3}_{3}=\rho-\tfrac{1}{6}R \quad (2.11b)$$

$$\mathbb{E}^{0}_{1}=0, \qquad \mathbb{E}^{0}_{2}=-\tfrac{1}{4}\nabla_{\alpha}S^{\alpha 0}_{2}=0, \qquad \mathbb{E}^{0}_{3}=0 \quad (2.11c)$$

$$\mathbb{E}^{1}_{2}=-\tfrac{1}{4}\nabla_{\alpha}S^{\alpha 1}_{2}=-\mathbb{E}^{1}_{2} \quad (2.11d)$$

$$\mathbb{E}^{3}_{2}=-\mathbb{E}^{2}_{3}=0 \quad (2.11e)$$

$$\mathbb{E}^{2}_{1}=-\mathbb{E}^{2}_{1}=-\tfrac{1}{4}\nabla_{\alpha}S^{\alpha 2}_{1} \quad (2.11f)$$

From equations (2.11) one immediately sees that although the properties $\mathbb{E}_{\mu\nu}g^{\mu\nu}=0$ and $\mathbb{H}_{\mu\nu}V^{v}=0$ are still valid in V_4, the symmetry property $\mathbb{E}_{\mu\nu}=\mathbb{E}_{\nu\mu}$ is not valid anymore in U_4. The presence of torsion introduces a skew symmetry in the quasi-Maxwellian fields. It is trivial to check that the same happens with the magnetic field $\mathbb{H}_{\mu\nu}$. From expression (2.10) one can also construct the quasi-Maxwellian invariants $I_1=C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ and $I_1=C_{\mu\nu\rho\gamma}*C^{\mu\nu\rho\sigma}$. This is done elsewhere (Garcia de Andrade, 1988).

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